manuscripta math. 124, 319–342 (2007)

Robert Denk · Thomas Krainer

$\mathcal{R}$-boundedness, pseudodifferential operators, and maximal regularity for some classes of partial differential operators

Received: 26 July 2006
Published online: 26 September 2007

Abstract. It is shown that an elliptic scattering operator A on a compact manifold with boundary with operator valued coefficients in the morphisms of a bundle of Banach spaces of class $(\mathcal{HT})$ and Pisier's property (α) has maximal regularity (up to a spectral shift), provided that the spectrum of the principal symbol of A on the scattering cotangent bundle avoids the right half-plane. This is accomplished by representing the resolvent in terms of pseudodifferential operators with $\mathcal{R}$-bounded symbols, yielding by an iteration argument the $\mathcal{R}$-boundedness of $\lambda(A-\lambda)^{-1}$ in $\Re(\lambda) \geq \gamma$ for some $\gamma \in \mathbb{R}$. To this end, elements of a symbolic and operator calculus of pseudodifferential operators with $\mathcal{R}$-bounded symbols are introduced. The significance of this method for proving maximal regularity results for partial differential operators is underscored by considering also a more elementary situation of anisotropic elliptic operators on $\mathbb{R}^d$ with operator valued coefficients.

1. Introduction

A central question in the analysis of parabolic evolution equations is whether a linear operator enjoys the property of maximal regularity. An account on how maximal regularity can be applied to partial differential equations is given in the survey paper by Prüss [14], and a general overview on developments in operator theory that are connected with maximal regularity can be found in Kunstmann and Weis [9], and the monograph [4] by Denk, Hieber, and Prüss.

Definition 1.1. A closed and densely defined operator A with domain $\mathcal{D}(A)$ in a Banach space X is said to have *maximal L_p-regularity*, if the associated evolution equation induces an isomorphism

$$\frac{d}{dt} - A : \mathring{W}_p^1([0,\infty), X) \cap L_p((0,\infty), \mathcal{D}(A)) \to L_p((0,\infty), X) \qquad (1.2)$$

for some $1 < p < \infty$. Here $\mathring{W}_p^1([0,\infty), X)$ consists of all $u \in L_p(\mathbb{R}, X)$ with $u' \in L_p(\mathbb{R}, X)$, and u supported in $[0,\infty)$.

R. Denk: Fachbereich Mathematik und Statistik, Fach D 193, Universität Konstanz, 78457 Konstanz, Germany

T. Krainer (✉): Mathematics and Statistics, Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601, USA. e-mail: krainer@psu.edu

Mathematics Subject Classification (2000): Primary 35K40; Secondary 35K65, 35R20, 58J05

DOI: 10.1007/s00229-007-0131-1

We shall assume here and in the sequel that the Banach space X if of class $(\mathcal{HT})$, i.e., the vector valued Hilbert transform is assumed to be continuous on $L_p(\mathbb{R}, X)$ (see also Definition 3.15, more on such spaces can be found in [1, 4]). It is known that the condition of maximal regularity does not depend on p, i.e., (1.2) is an isomorphism for all $1 < p < \infty$ once that this is the case for some p.

If the Banach space X is an L_q-space, an application of the closed graph theorem to (1.2) reveals that the property of maximal regularity is connected with proving optimal apriori L_p–L_q estimates for solutions. This gives a hint why maximal regularity is important in the theory of nonlinear partial differential equations, because apriori estimates of such kind for the linearized equation and a contraction principle readily imply local existence of solutions to the nonlinear equation. Recent work on optimal L_p–L_q estimates for parabolic evolution equations that rely on heat kernel estimates includes [10], see also [7].

There are several approaches to prove maximal regularity for a given operator A. One approach is to check whether A admits a bounded H^∞-calculus (or merely bounded imaginary powers). A famous result by Dore and Venni [5] then implies maximal regularity. Another way involves only the resolvent of A and relies on operator valued Fourier multiplier theorems due to Weis [18]: A has maximal L_p-regularity provided that $A - \lambda : \mathcal{D}(A) \to X$ is invertible for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) \geq 0$, and the resolvent $\{\lambda(A-\lambda)^{-1};\ \Re(\lambda) \geq 0\} \subset \mathscr{L}(X)$ is $\mathcal{R}$-bounded (see Sect. 2 further below for the definition and properties of $\mathcal{R}$-bounded sets of operators).

For elliptic partial differential operators A, the way to get a hold on its powers is via making use of the resolvent and a Seeley theorem, i.e., a representation of $(A - \lambda)^{-1}$ in terms of pseudodifferential operators (see the classical paper [15], or [16]). However, in the case of operators on noncompact and singular manifolds, it may still be quite difficult to actually pass from a corresponding Seeley theorem to the powers (or H^∞-calculus) of A. The way to prove maximal regularity by establishing the $\mathcal{R}$-boundedness of resolvents seems to be more direct for differential operators. The intuition is that a Seeley theorem alone should already be sufficient.

In this work, we follow the latter philosophy. We investigate pseudodifferential operators depending on parameters that play the role of the spectral parameter, and we prove that they give rise to families of continuous operators in Sobolev spaces that are $\mathcal{R}$-bounded with suitable bounds (see Theorems 3.18 and 5.3). Hence the representation of the resolvent in terms of a pseudodifferential parametrix that depends on the spectral parameter immediately yields the desired $\mathcal{R}$-boundedness, thus maximal regularity.

To achieve this, we introduce and investigate in Sects. 2 and 3 some properties of a symbolic calculus with operator valued symbols that satisfy symbol estimates in terms of $\mathcal{R}$-bounds rather than operator norm bounds, and we investigate associated pseudodifferential operators. Surprisingly for us, not much seems to have been done in this direction (see [8, 13, 17]). One observation in this context is that classical operator valued symbols that are modeled on the operator norm (i.e., symbols that admit asymptotic expansions with homogeneous components) automatically satisfy the strong $\mathcal{R}$-bounded symbol estimates (Proposition 3.10). This is important since classical symbols are the ones that appear naturally when constructing parametrices for differential operators. Theorem 3.18 on families of pseudodifferential operators can be interpreted as an iteration result: A pseudodifferential operator depending on parameters with $\mathcal{R}$-bounded symbol induces a family of continuous operators in Sobolev spaces that satisfies suitable $\mathcal{R}$-bounds. More precisely, this family is itself an $\mathcal{R}$-bounded operator valued symbol depending on the parameter.

In Sect. 4 we illustrate the proposed method by proving the $\mathcal{R}$-boundedness of resolvents of parameter-dependent anisotropic elliptic operators A on $\mathbb{R}^d$ in anisotropic Sobolev spaces (Theorem 4.3). A parameter-dependent parametrix of $A - \lambda$ can simply be constructed by the standard method of symbolic inversion and a formal Neumann series argument, and the results from Sects. 2 and 3 give the desired conclusion.

Finally, in Sect. 5, we consider the more advanced situation of elliptic scattering operators A on manifolds with boundary (see Melrose [11]). Assuming the appropriate ellipticity condition on the principal scattering symbol of A we show that the resolvent is an $\mathcal{R}$-bounded family in the scattering Sobolev spaces. This is again achieved directly from a Seeley theorem, i.e., the resolvent $(A-\lambda)^{-1}$ is represented in terms of a parameter-dependent parametrix in the scattering pseudodifferential calculus (Theorem 5.3).

In both Sects. 4 and 5 the differential operator A in question is admitted to have operator valued coefficients. The underlying Banach space (or bundle) is assumed to be of class $(\mathcal{HT})$ and to have Pisier's property (α) (see, e.g., [3, 12]). Besides of the observation that in this way the case of systems with infinitely many equations and unknowns is included, operators of such kind have recently been investigated in coagulation and fragmentation models (see Amann [2], though, as pointed out in subsequent work by Amann, in many cases one is interested in operator valued coefficients acting on L_1-spaces that fail to be of class $(\mathcal{HT})$).

2. Preliminaries on $\mathcal{R}$-boundedness

Definition 2.1. Let X and Y be Banach spaces. A subset $\mathcal{T} \subset \mathscr{L}(X, Y)$ is called *$\mathcal{R}$-bounded*, if for some $1 \le p < \infty$ and some constant $C_p \ge 0$ the inequality

$$\left(\sum_{\varepsilon_1,\dots,\varepsilon_N \in \{-1,1\}} \left\| \sum_{j=1}^N \varepsilon_j T_j x_j \right\|^p \right)^{1/p} \le C_p \left(\sum_{\varepsilon_1,\dots,\varepsilon_N \in \{-1,1\}} \left\| \sum_{j=1}^N \varepsilon_j x_j \right\|^p \right)^{1/p} \tag{2.2}$$

holds for all choices of $T_1, \dots, T_N \in \mathcal{T}$ and $x_1, \dots, x_N \in X$, $N \in \mathbb{N}$.

The best constant

$$C_p = \sup \left\{ \left(\sum_{\varepsilon_1,\dots,\varepsilon_N \in \{-1,1\}} \left\| \sum_{j=1}^N \varepsilon_j T_j x_j \right\|^p \right)^{1/p} ; \ N \in \mathbb{N},\ T_1, \dots, T_N \in \mathcal{T}, \right.$$
$$\left. \left(\sum_{\varepsilon_1,\dots,\varepsilon_N \in \{-1,1\}} \left\| \sum_{j=1}^N \varepsilon_j x_j \right\|^p \right)^{1/p} = 1 \right\}$$

in (2.2) is called the *$\mathcal{R}$-bound* of $\mathcal{T}$ and will be denoted by $\mathcal{R}(\mathcal{T})$. By *Kahane's inequality* (2.4) the notion of $\mathcal{R}$-boundedness is independent of $1 \leq p < \infty$, and the $\mathcal{R}$-bounds for different values of p are equivalent which is the justification for suppressing p from the notation.

Remark 2.3. The following results related to $\mathcal{R}$-bounded sets in $\mathscr{L}(X, Y)$ are well established in the literature, see [4, 9] and the references given there.

(i) For Hilbert spaces X and Y the notion of $\mathcal{R}$-boundedness reduces merely to boundedness.

(ii) *Kahane's inequality*: For every Banach space X and all values $1 \leq p, q < \infty$ there exist constants $c,\ C > 0$ such that

$$c\left(2^{-N}\sum_{\varepsilon_1,\ldots,\varepsilon_N\in\{-1,1\}}\left\|\sum_{j=1}^N \varepsilon_j x_j\right\|^q\right)^{1/q} \leq \left(2^{-N}\sum_{\varepsilon_1,\ldots,\varepsilon_N\in\{-1,1\}}\left\|\sum_{j=1}^N \varepsilon_j x_j\right\|^p\right)^{1/p}$$
$$\leq C\left(2^{-N}\sum_{\varepsilon_1,\ldots,\varepsilon_N\in\{-1,1\}}\left\|\sum_{j=1}^N \varepsilon_j x_j\right\|^q\right)^{1/q} \tag{2.4}$$

for all choices $x_1, \ldots, x_N \in X$, $N \in \mathbb{N}$.

(iii) *Kahane's contraction principle*: For all $\alpha_j,\ \beta_j \in \mathbb{C}$ with $|\alpha_j| \leq |\beta_j|$, $j = 1, \ldots, N$, the inequality

$$\left(\sum_{\varepsilon_1,\ldots,\varepsilon_N\in\{-1,1\}}\left\|\sum_{j=1}^N \varepsilon_j \alpha_j x_j\right\|^p\right)^{1/p} \leq 2\left(\sum_{\varepsilon_1,\ldots,\varepsilon_N\in\{-1,1\}}\left\|\sum_{j=1}^N \varepsilon_j \beta_j x_j\right\|^p\right)^{1/p} \tag{2.5}$$

holds for all choices $x_1, \ldots, x_N \in X$, $N \in \mathbb{N}$.
In particular, the set $\{\lambda I;\ |\lambda| \leq R\} \subset \mathscr{L}(X)$ is $\mathcal{R}$-bounded for every $R > 0$.

(iv) For $\mathcal{T},\ \mathcal{S} \subset \mathscr{L}(X, Y)$ we have

$$\mathcal{R}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}(\mathcal{T}) + \mathcal{R}(\mathcal{S}). \tag{2.6}$$

(v) For Banach spaces $X,\ Y,\ Z$ and $\mathcal{T} \subset \mathscr{L}(Y, Z)$, $\mathcal{S} \subset \mathscr{L}(X, Y)$ we have

$$\mathcal{R}(\mathcal{T}\mathcal{S}) \leq \mathcal{R}(\mathcal{T})\mathcal{R}(\mathcal{S}). \tag{2.7}$$

(vi) Let $\mathcal{T} \subset \mathscr{L}(X, Y)$, and let $\overline{\mathrm{aco}(\mathcal{T})}$ be the closure of the absolute convex hull of $\mathcal{T}$ in the strong operator topology. Then

$$\mathcal{R}\left(\overline{\mathrm{aco}(\mathcal{T})}\right) \leq 2\mathcal{R}(\mathcal{T}). \tag{2.8}$$

Definition 2.9. Let Γ be a set. Define $\ell^\infty_{\mathcal{R}}(\Gamma, \mathscr{L}(X, Y))$ as the space of all functions $f : \Gamma \to \mathscr{L}(X, Y)$ with $\mathcal{R}$-bounded range and norm

$$\|f\|_{\ell^\infty_{\mathcal{R}}} := \mathcal{R}\left(f(\Gamma)\right). \tag{2.10}$$

Proposition 2.11. $\left(\ell^\infty_{\mathcal{R}}(\Gamma, \mathscr{L}(X,Y)), \|\cdot\|_{\ell^\infty_{\mathcal{R}}}\right)$ *is a Banach space. The embedding*

$$\ell^\infty_{\mathcal{R}}(\Gamma, \mathscr{L}(X,Y)) \hookrightarrow \ell^\infty(\Gamma, \mathscr{L}(X,Y))$$

into the Banach space $\ell^\infty(\Gamma, \mathscr{L}(X,Y))$ *of all* $\mathscr{L}(X,Y)$*-valued functions on* Γ *with bounded range is a contraction.*

The norm in $\ell^\infty_{\mathcal{R}}$ *is submultiplicative, i.e.,*

$$\|f \cdot g\|_{\ell^\infty_{\mathcal{R}}} \le \|f\|_{\ell^\infty_{\mathcal{R}}} \cdot \|g\|_{\ell^\infty_{\mathcal{R}}}$$

whenever the composition $f \cdot g$ *makes sense, and we have* $\|\mathbb{1}\|_{\ell^\infty_{\mathcal{R}}} = 1$ *for the constant map* $\mathbb{1} \equiv \mathrm{Id}_X$.

Proof. Definiteness and homogeneity of the norm (2.10) are immediate consequences of Definition 2.1. The triangle inequality follows from (2.6), the submultiplicativity from (2.7). Moreover, the embedding $\ell^\infty_{\mathcal{R}} \hookrightarrow \ell^\infty$ is a contraction because the $\mathcal{R}$-bound of a set is always greater or equal to its operator norm bound.

It remains to show completeness. Let $(f_j)_j \subset \ell^\infty_{\mathcal{R}}$ be a Cauchy sequence. Thus $(f_j)_j$ is also a Cauchy sequence in $\ell^\infty(\Gamma, \mathscr{L}(X,Y))$, and there exists $f \in \ell^\infty(\Gamma, \mathscr{L}(X,Y))$ with $\|f_j - f\|_{\ell^\infty} \to 0$ as $j \to \infty$. Let $\varepsilon > 0$, and let $N(\varepsilon) \in \mathbb{N}$ be such that $\|f_j - f_k\|_{\ell^\infty_{\mathcal{R}}} \le \varepsilon$ for $j,k \ge N(\varepsilon)$. In view of Definition 2.1 this implies that

$$\left(\sum_{\varepsilon_1,\dots,\varepsilon_N \in \{-1,1\}} \left\| \sum_{i=1}^N \varepsilon_i (f_j(\gamma_i)x_i - f_k(\gamma_i)x_i) \right\|^p \right)^{1/p} \le \varepsilon \tag{2.12}$$

for all finite collections $x_1, \dots, x_N \in X$ with $(\sum_{\varepsilon_1,\dots,\varepsilon_N \in \{-1,1\}} \|\sum_{j=1}^N \varepsilon_i x_j\|^p)^{1/p} = 1$, and all choices $\gamma_1, \dots, \gamma_N \in \Gamma$. Letting $k \to \infty$ in (2.12) gives

$$\left(\sum_{\varepsilon_1,\dots,\varepsilon_N \in \{-1,1\}} \left\| \sum_{i=1}^N \varepsilon_i (f_j(\gamma_i)x_i - f(\gamma_i)x_i) \right\|^p \right)^{1/p} \le \varepsilon,$$

and passing to the supremum over all possible choices implies $\|f_j - f\|_{\ell^\infty_{\mathcal{R}}} \le \varepsilon$ for $j \ge N(\varepsilon)$. This shows $f \in \ell^\infty_{\mathcal{R}}$ and $f_j \to f$ with respect to $\|\cdot\|_{\ell^\infty_{\mathcal{R}}}$, and the proof is complete. □

Proposition 2.13. *We have*

$$\ell^\infty(\Gamma) \hat{\otimes}_\pi \mathscr{L}(X,Y) \subset \ell^\infty_{\mathcal{R}}(\Gamma, \mathscr{L}(X,Y)).$$

Recall that $\ell^\infty(\Gamma)\hat{\otimes}_\pi \mathscr{L}(X,Y)$ *is realized as the space of all functions* $f : \Gamma \to \mathscr{L}(X,Y)$ *that can be represented as*

$$f(\gamma) = \sum_{j=1}^\infty \lambda_j f_j(\gamma) A_j$$

with sequences $(\lambda_j)_j \in \ell^1(\mathbb{N})$*, and* $f_j \to 0$ *in* $\ell^\infty(\Gamma)$ *as well as* $A_j \to 0$ *in* $\mathscr{L}(X,Y)$.

Proof. For $j \in \mathbb{N}$ the function $\lambda_j f_j \otimes A_j : \Gamma \to \mathscr{L}(X,Y)$ belongs to $\ell^\infty_{\mathcal{R}}(\Gamma, \mathscr{L}(X,Y))$ with norm

$$\|\lambda_j f_j \otimes A_j\|_{\ell^\infty_{\mathcal{R}}} \leq 2 \cdot |\lambda_j| \cdot \|f_j\|_{\ell^\infty(\Gamma)} \cdot \|A_j\|_{\mathscr{L}(X,Y)}$$

in view of Kahane's contraction principle (2.5). Consequently,

$$\sum_{j=1}^{\infty} \|\lambda_j f_j \otimes A_j\|_{\ell^\infty_{\mathcal{R}}} < \infty,$$

and by completeness the series $f = \sum_{j=1}^{\infty} \lambda_j f_j \otimes A_j$ converges in $\ell^\infty_{\mathcal{R}}(\Gamma, \mathscr{L}(X,Y))$. □

As a consequence of Proposition 2.13 we obtain the following corollary about the $\mathcal{R}$-boundedness of the range of certain very regular functions. Actually, much less regularity is necessary to draw this conclusion, see, e.g., [6]. However, for our purposes the corollary is sufficient, and its proof is based on elementary arguments.

Corollary 2.14. (i) *Let M be a smooth manifold, $K \subset M$ a compact subset, and $f \in C^\infty(M, \mathscr{L}(X,Y))$. Then $f(K)$ is an $\mathcal{R}$-bounded subset of $\mathscr{L}(X,Y)$.*
(ii) *Let $f \in \mathscr{S}(\mathbb{R}^n, \mathscr{L}(X,Y))$. Then the range $f(\mathbb{R}^n) \subset \mathscr{L}(X,Y)$ is $\mathcal{R}$-bounded.*

Proof. The assertion follows from Proposition 2.13 in view of

$$C^\infty(M, \mathscr{L}(X,Y)) \cong C^\infty(M)\hat{\otimes}_\pi \mathscr{L}(X,Y),$$
$$\mathscr{S}(\mathbb{R}^n, \mathscr{L}(X,Y)) \cong \mathscr{S}(\mathbb{R}^n)\hat{\otimes}_\pi \mathscr{L}(X,Y).$$

□

3. Operator valued $\mathcal{R}$-bounded symbols, and pseudodifferential operators on $\mathbb{R}^d$

In this section we consider special classes of anisotropic operator valued symbols and associated pseudodifferential operators in $\mathbb{R}^d$ depending on parameters. In what follows, let $n \in \mathbb{N}$ be total dimension of parameters and covariables. Throughout this section we fix a vector $\vec{\ell} = (\ell_1, \dots, \ell_n) \in \mathbb{N}^n$ of positive integers which represents the anisotropy. For $\xi \in \mathbb{R}^n$ we denote

$$|\xi|_{\vec{\ell}} = \left(\sum_{j=1}^{n} \xi_j^{2\pi_j}\right)^{\frac{1}{2\ell_1\cdots\ell_n}}, \quad \langle\xi\rangle_{\vec{\ell}} = \left(1 + \sum_{j=1}^{n} \xi_j^{2\pi_j}\right)^{\frac{1}{2\ell_1\cdots\ell_n}}, \quad \text{where } \pi_j = \prod_{i\neq j} \ell_i,$$

and for a multi-index $\beta \in \mathbb{N}_0^n$ let $|\beta|_{\vec{\ell}} = \sum_{j=1}^{n} \ell_j \beta_j$ be its anisotropic length. We are aware that the notation $|\cdot|_{\vec{\ell}}$ is ambiguous, on the other hand multi-indices and covectors (or parameters) are easily distinguishable by the context where they appear.

Apparently, Peetre's inequality

$$\langle \xi + \xi' \rangle_{\vec{\ell}}^{s} \leq 2^{|s|} \langle \xi \rangle_{\vec{\ell}}^{s} \cdot \langle \xi' \rangle_{\vec{\ell}}^{|s|}$$

holds for all $s \in \mathbb{R}$, and there exist constants $c, C > 0$ depending only on $\vec{\ell}$ and n such that

$$c\langle \xi \rangle^{1/\sum_{j=1}^{n} \ell_j} \leq \langle \xi \rangle_{\vec{\ell}} \leq C\langle \xi \rangle^{\sum_{j=1}^{n} 1/\ell_j},$$

where as usual $\langle \xi \rangle = \left(1 + |\xi|^2\right)^{1/2}$ is the standard regularized distance function.

Let X and Y be Banach spaces, $\mu \in \mathbb{R}$, and let $S^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y)$ denote the standard space of anisotropic $\mathscr{L}(X, Y)$-valued symbols on $\mathbb{R}^n$, i.e., the space of all $a \in C^\infty(\mathbb{R}^n, \mathscr{L}(X, Y))$ such that

$$\sup_{\xi \in \mathbb{R}^n} \| \langle \xi \rangle_{\vec{\ell}}^{-\mu + |\beta|_{\vec{\ell}}} \partial_\xi^\beta a(\xi) \|_{\mathscr{L}(X,Y)} < \infty$$

for all $\beta \in \mathbb{N}_0^n$.

We shall be mainly concerned with the following more restrictive symbol class:

Definition 3.1. A function $a \in C^\infty(\mathbb{R}^n, \mathscr{L}(X, Y))$ belongs to $S_{\mathcal{R}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y)$ if and only if for all $\beta \in \mathbb{N}_0^n$

$$|a|_\beta^{(\mu;\vec{\ell})} := \mathcal{R}\left(\left\{ \langle \xi \rangle_{\vec{\ell}}^{-\mu+|\beta|_{\vec{\ell}}} \partial_\xi^\beta a(\xi);\ \xi \in \mathbb{R}^n \right\} \right) < \infty. \tag{3.2}$$

By Proposition 2.11 and the usual arguments we obtain that $S_{\mathcal{R}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y)$ is a Fréchet space in the topology generated by the seminorms $|\cdot|_\beta^{(\mu;\vec{\ell})}$.

Lemma 3.3. (i) *For $\beta \in \mathbb{N}_0^n$ differentiation*

$$\partial_\xi^\beta : S_{\mathcal{R}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y) \to S_{\mathcal{R}}^{\mu-|\beta|_{\vec{\ell}};\vec{\ell}}(\mathbb{R}^n; X, Y)$$

is continuous.

(ii) *The embedding*

$$S_{\mathcal{R}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y) \hookrightarrow S_{\mathcal{R}}^{\mu';\vec{\ell}}(\mathbb{R}^n; X, Y)$$

is continuous for $\mu \leq \mu'$.

(iii) *For Banach spaces X, Y, and Z the multiplication (pointwise composition)*

$$S_{\mathcal{R}}^{\mu;\vec{\ell}}(\mathbb{R}^n; Y, Z) \times S_{\mathcal{R}}^{\mu';\vec{\ell}}(\mathbb{R}^n; X, Y) \to S_{\mathcal{R}}^{\mu+\mu';\vec{\ell}}(\mathbb{R}^n; X, Z)$$

is bilinear and continuous.

(iv) *The embedding*

$$S_{\mathcal{R}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y) \hookrightarrow S^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y)$$

into the standard class of anisotropic operator valued symbols is continuous.

(v) *The space of scalar symbols* $S^{\mu;\vec{\ell}}(\mathbb{R}^n)$ *embeds into* $S^{\mu;\vec{\ell}}_{\mathcal{R}}(\mathbb{R}^n; X, X)$ *via* $a(\xi) \mapsto a(\xi)\mathrm{Id}_X$.

Proof. (i) and (iv) are evident, (ii) and (v) follow from Kahane's contraction principle (2.5), and (iii) is straightforward from the Leibniz rule and Proposition 2.11 (submultiplicativity and subadditivity of the norm $\|\cdot\|_{\ell^\infty_{\mathcal{R}}}$). □

Lemma 3.4. *We have* $S^{-\infty}_{\mathcal{R}}(\mathbb{R}^n; X, Y) = S^{-\infty}(\mathbb{R}^n; X, Y)$*, i.e.,*

$$\bigcap_{\mu\in\mathbb{R}} S^{\mu;\vec{\ell}}_{\mathcal{R}}(\mathbb{R}^n; X, Y) = \bigcap_{\mu\in\mathbb{R}} S^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y), \tag{3.5}$$

and this space does not depend on the anisotropy $\vec{\ell}$.

Proof. By Lemma 3.3 we have $S^{-\infty}_{\mathcal{R}} \subset S^{-\infty}$. On the other hand, we may write

$$S^{-\infty}(\mathbb{R}^n; X, Y) \cong \mathscr{S}(\mathbb{R}^n)\hat{\otimes}_\pi \mathscr{L}(X, Y),$$

and thus we obtain $S^{-\infty} \subset S^{-\infty}_{\mathcal{R}}$ from Proposition 2.13. □

Definition 3.6. Let $a_j \in S^{\mu_j;\vec{\ell}}_{\mathcal{R}}(\mathbb{R}^n; X, Y)$ with $\mu_j \to -\infty$, and let $\overline{\mu} = \max \mu_j$. For a symbol $a \in S^{\overline{\mu};\vec{\ell}}_{\mathcal{R}}(\mathbb{R}^n; X, Y)$ write $a \underset{\mathcal{R}}{\sim} \sum_{j=1}^\infty a_j$ if for all $K \in \mathbb{R}$ there exists $N(K)$ such that

$$a - \sum_{j=1}^N a_j \in S^{K;\vec{\ell}}_{\mathcal{R}}(\mathbb{R}^n; X, Y)$$

for $N > N(K)$.

Recall that the standard notion of asymptotic expansion $a \sim \sum_{j=1}^\infty a_j$ means that for all $K \in \mathbb{R}$ there exists $N(K)$ such that

$$a - \sum_{j=1}^N a_j \in S^{K;\vec{\ell}}(\mathbb{R}^n; X, Y)$$

for $N > N(K)$.

Proposition 3.7. *Let* $a_j \in S^{\mu_j;\vec{\ell}}_{\mathcal{R}}(\mathbb{R}^n; X, Y)$ *with* $\mu_j \to -\infty$*, and let* $\overline{\mu} = \max \mu_j$*. Then there exists* $a \in S^{\overline{\mu};\vec{\ell}}_{\mathcal{R}}(\mathbb{R}^n; X, Y)$ *with* $a \underset{\mathcal{R}}{\sim} \sum_{j=1}^\infty a_j$.

Proof. The proof is based on the usual Borel argument. Let $\chi \in C^\infty(\mathbb{R}^n)$ be a function with $\chi \equiv 0$ near the origin and $\chi \equiv 1$ near infinity, and define

$$\chi_\theta(\xi) := \chi\left(\frac{\xi_1}{\theta^{\ell_1}}, \dots, \frac{\xi_n}{\theta^{\ell_n}}\right)$$

for $\theta \geq 1$. Then the family $\{\chi_\theta;\ \theta \geq 1\} \subset S^{0;\vec{\ell}}(\mathbb{R}^n)$ is bounded. To see this assume that $\chi \equiv 1$ for $|\xi|_{\vec{\ell}} \geq c$ and $\chi \equiv 0$ for $|\xi|_{\vec{\ell}} \leq \frac{1}{c}$. Then, for $\beta \neq 0$,

$$\partial_\xi^\beta \chi_\theta(\xi) = \theta^{-|\beta|_{\vec{\ell}}} \left(\partial_\xi^\beta \chi\right) \left(\frac{\xi_1}{\theta^{\ell_1}}, \ldots, \frac{\xi_n}{\theta^{\ell_n}}\right) \neq 0$$

at most for $\frac{1}{c}|\xi|_{\vec{\ell}} \leq \theta \leq c|\xi|_{\vec{\ell}}$.

Let $p \in S_{\mathcal{R}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y)$, $\mu \in \mathbb{R}$, and define $p_\theta(\xi) = \chi_\theta(\xi)p(\xi)$, $\theta \geq 1$. By Lemma 3.3 the set $\{p_\theta;\ \theta \geq 1\} \subset S_{\mathcal{R}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y)$ is bounded. Let $\mu' > \mu$ and $\varepsilon > 0$. Fix $R(\varepsilon) \geq 1$ such that for $\theta \geq R(\varepsilon)$ we have $\chi_\theta(\xi) \neq 0$ at most for $\xi \in \mathbb{R}^n$ with $\langle\xi\rangle_{\vec{\ell}}^{\mu'-\mu} \geq \frac{2}{\varepsilon}$. Consequently, for such θ we obtain from Kahane's contraction principle (2.5)

$$\begin{aligned}
&\left(\sum_{\varepsilon_1,\ldots,\varepsilon_N \in\{-1,1\}} \left\| \sum_{j=1}^N \varepsilon_j \langle\xi_j\rangle_{\vec{\ell}}^{-\mu'+|\beta|_{\vec{\ell}}} \partial_\xi^\beta p_\theta(\xi_j) x_j \right\|^p \right)^{1/p} \\
&\leq \varepsilon \left(\sum_{\varepsilon_1,\ldots,\varepsilon_N \in\{-1,1\}} \left\| \sum_{j=1}^N \varepsilon_j \langle\xi_j\rangle_{\vec{\ell}}^{-\mu+|\beta|_{\vec{\ell}}} \partial_\xi^\beta p_\theta(\xi_j) x_j \right\|^p \right)^{1/p} \\
&\leq \varepsilon |p_\theta|_\beta^{(\mu;\vec{\ell})} \left(\sum_{\varepsilon_1,\ldots,\varepsilon_N \in\{-1,1\}} \left\| \sum_{j=1}^N \varepsilon_j x_j \right\|^p \right)^{1/p}
\end{aligned}$$

which shows that $|p_\theta|_\beta^{(\mu';\vec{\ell})} \leq \varepsilon |p_\theta|_\beta^{(\mu;\vec{\ell})}$ for $\theta \geq R(\varepsilon)$. The boundedness of $\{p_\theta;\ \theta \geq 1\}$ in $S_{\mathcal{R}}^{\mu;\vec{\ell}}$ now implies that $p_\theta \to 0$ in $S_{\mathcal{R}}^{\mu';\vec{\ell}}$ as $\theta \to \infty$.

Now we proceed with the construction of the symbol a. We may assume without loss of generality that $\mu_j > \mu_{j+1}$ for $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ let $p_1^j \leq p_2^j \leq \ldots$ be an increasing fundamental system of seminorms for the topology of $S_{\mathcal{R}}^{\mu_j;\vec{\ell}}$. Pick a sequence $1 \leq c_k^1 < c_{k+1}^1 \to \infty$ such that $p_k^1(a_{k,\theta}) < 2^{-k}$ for $k > 1$ and all $\theta \geq c_k^1$. We proceed by induction and construct successively subsequences $(c_k^j)_k$ of $(c_k^{j-1})_k$ such that $p_k^j(a_{k,\theta}) < 2^{-k}$ for $k > j$ and all $\theta \geq c_k^j$. Let $c_k := c_k^k$ be the diagonal sequence. Then $p_k^j(a_{k,c_k}) < 2^{-k}$ for $k > j$ which shows that the series $\sum_{k=j}^\infty a_{k,c_k}$ is unconditionally convergent in $S_{\mathcal{R}}^{\mu_j;\vec{\ell}}$. Now let $a := \sum_{k=1}^\infty a_{k,c_k}$. Then

$$a - \sum_{k=1}^N a_k = \sum_{k=N+1}^\infty a_{k,c_k} + \sum_{k=1}^N (a_{k,c_k} - a_k).$$

As every summand $a_{k,c_k} - a_k$ is compactly supported in ξ, we obtain from Lemma 3.4 that

$$\sum_{k=1}^N (a_{k,c_k} - a_k) \in S_{\mathcal{R}}^{-\infty}(\mathbb{R}^n; X, Y),$$

and consequently $a - \sum_{k=1}^N a_k \in S_{\mathcal{R}}^{\mu_{N+1};\vec{\ell}}$ as desired. □

The combination of the possibility to carry out asymptotic expansions within the classes $S_{\mathcal{R}}^{\mu;\vec{\ell}}$ and the identity $S^{-\infty} = S_{\mathcal{R}}^{-\infty}$ are very useful for proving that the $\mathcal{R}$-bounded symbol classes remain preserved under manipulations of the symbolic calculus. Let us formulate this more precisely:

Lemma 3.8. *Let* $\overline{\mu} = \max_{j=1}^{\infty} \mu_j$, *where* $\mu_j \to -\infty$ *as* $j \to \infty$, *and let* $a \in S^{\overline{\mu};\vec{\ell}}$ *with* $a \sim \sum_{j=1}^{\infty} a_j$. *Suppose that the summands in the asymptotic expansion satisfy* $a_j \in S_{\mathcal{R}}^{\mu_j;\vec{\ell}}$ *for every* $j \in \mathbb{N}$. *Then* $a \in S_{\mathcal{R}}^{\overline{\mu};\vec{\ell}}$, *and* $a \underset{\mathcal{R}}{\sim} \sum_{j=1}^{\infty} a_j$.

Proof. According to Proposition 3.7 there exists a symbol $c \in S_{\mathcal{R}}^{\overline{\mu};\vec{\ell}}$ with $c \underset{\mathcal{R}}{\sim} \sum_{j=1}^{\infty} a_j$. In particular we have $c \sim \sum_{j=1}^{\infty} a_j$, and consequently $a - c \in S^{-\infty} = S_{\mathcal{R}}^{-\infty}$ by Lemma 3.4. Thus $a \in S_{\mathcal{R}}^{\overline{\mu};\vec{\ell}}$ with $a \underset{\mathcal{R}}{\sim} c \underset{\mathcal{R}}{\sim} \sum_{j=1}^{\infty} a_j$ as desired. □

We are going to make use of Lemma 3.8 to show that the standard classes of classical anisotropic symbols are a subspace of the $\mathcal{R}$-bounded ones, see Proposition 3.10 further below. To this end, recall the definition of the spaces of classical symbols:

Definition 3.9. (i) A function $a \in C^{\infty}(\mathbb{R}^n \setminus \{0\}, \mathscr{L}(X, Y))$ is called *(anisotropic) homogeneous of degree* $\mu \in \mathbb{R}$ iff

$$a\left(\varrho^{\ell_1}\xi_1, \ldots, \varrho^{\ell_n}\xi_n\right) = \varrho^{\mu} a(\xi)$$

for $\varrho > 0$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.

(ii) The space $S_{\mathrm{cl}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y)$ of *classical symbols* of order μ consists of all $a \in S^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y)$ such that there exists a sequence $a_{(\mu-j)}$ of anisotropic homogeneous functions of degree $\mu - j$, $j \in \mathbb{N}_0$, such that for some excision function $\chi \in C^{\infty}(\mathbb{R}^n)$ with $\chi \equiv 0$ near the origin and $\chi \equiv 1$ near infinity $a(\xi) \sim \sum_{j=0}^{\infty} \chi(\xi) a_{(\mu-j)}(\xi)$.

Recall that $S_{\mathrm{cl}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y)$ is a Fréchet space in the projective topology with respect to the mappings

$$a \mapsto a(\xi) - \sum_{j=0}^{N} \chi(\xi) a_{(\mu-j)}(\xi) \in S^{\mu-N-1;\vec{\ell}}(\mathbb{R}^n; X, Y), \qquad N = -1, 0, 1, \ldots,$$

$$a \mapsto a_{(\mu-j)}|_{\{|\xi|_{\vec{\ell}}=1\}} \in C^{\infty}\left(\{|\xi|_{\vec{\ell}} = 1\}, \mathscr{L}(X, Y)\right), \qquad j = 0, 1, 2, \ldots.$$

Note that the function $\xi \mapsto \langle\xi\rangle_{\vec{\ell}}^{\mu}$ is a classical scalar symbol of order $\mu \in \mathbb{R}$.

Proposition 3.10. $S_{\mathrm{cl}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y) \hookrightarrow S_{\mathcal{R}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y)$.

Proof. Let $a \in S_{\mathrm{cl}}^{\mu;\vec{\ell}}(\mathbb{R}^n; X, Y)$. In view of $a(\xi) \sim \sum_{j=0}^{\infty} \chi(\xi) a_{(\mu-j)}(\xi)$ and Lemma 3.8 it suffices to show that each summand $\chi(\xi)a_{(\mu-j)}(\xi)$ in this asymptotic expansion belongs to $S_{\mathcal{R}}^{\mu-j;\vec{\ell}}(\mathbb{R}^n; X, Y)$.

Consider therefore the function $p(\xi) = \chi(\xi)a_{(\mu)}(\xi)$. Note that $\partial_\xi^\beta a_{(\mu)}(\xi)$ is anisotropic homogeneous of degree $\mu - |\beta|_{\vec{\ell}}$. Hence, by the Leibniz rule and the fact that compactly supported smooth functions in ξ belong to $S_{\mathcal{R}}^{-\infty}$ by Lemma 3.4, we conclude that it is sufficient to prove that the seminorm $|p|_0^{(\mu;\vec{\ell})} < \infty$, see (3.2). We have

$$\langle\xi\rangle_{\vec{\ell}}^{-\mu}\left(\chi(\xi)a_{(\mu)}(\xi)\right) = \chi(\xi)a\left(\frac{\xi_1}{\langle\xi\rangle_{\vec{\ell}}^{\ell_1}}, \dots, \frac{\xi_n}{\langle\xi\rangle_{\vec{\ell}}^{\ell_n}}\right),$$

and thus this operator family is $\mathcal{R}$-bounded for $\xi \in \mathbb{R}^n$ in view of Corollary 2.14. □

Remark 3.11. Let $\emptyset \neq \Lambda \subset \mathbb{R}^n$. The symbol space $S_{\mathcal{R}}^{\mu;\vec{\ell}}(\Lambda; X, Y)$ is defined as the space of restrictions of the $\mathcal{R}$-bounded symbol class on $\mathbb{R}^n$ to Λ endowed with the quotient topology, and analogously we define the classes of (classical) ordinary symbols on Λ. In this way, the results of this section carry over immediately to symbols on Λ.

Pseudodifferential operators. Let $n = d+q$, and we change the notation slightly so as to consider covariables $\xi \in \mathbb{R}^d$ and parameters $\lambda \in \Lambda \subset \mathbb{R}^q$. Let $\vec{\ell} = (\vec{\ell}', \vec{\ell}'') \in \mathbb{N}^{d+q}$ be the vector that determines the anisotropy of covariables and parameters.

With a symbol $a(x,\xi,\lambda) \in S_{\mathrm{cl}}^0(\mathbb{R}_x^d, S_{\mathrm{cl}}^{\mu;\vec{\ell}}(\mathbb{R}_\xi^d \times \Lambda; X, Y))$ we associate a family of pseudodifferential operators $\mathrm{op}_x(a)(\lambda) : \mathscr{S}(\mathbb{R}^d, X) \to \mathscr{S}(\mathbb{R}^d, Y)$ as usual via

$$\mathrm{op}_x(a)(\lambda)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi} a(x,\xi,\lambda)\hat{u}(\xi)\, d\xi,$$

where $\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-iy\xi}u(y)\,dy$ is the Fourier transform of the function $u \in \mathscr{S}(\mathbb{R}^d, X)$. Here and in what follows, much less requirements on the behavior at infinity of the x-dependence of the symbol $a(x,\xi,\lambda)$ are necessary. However, for our purposes this (rather strong) condition is sufficient, and the arguments become considerably simpler. We first observe the following

Lemma 3.12. *Let $a(x,\xi,\lambda) \in S_{\mathrm{cl}}^0(\mathbb{R}_x^d, S_{\mathrm{cl}}^{\mu;\vec{\ell}}(\mathbb{R}_\xi^d \times \Lambda; X, Y))$. Then for every α, β the set*

$$\left\{\langle x\rangle^{|\alpha|}\langle\xi,\lambda\rangle_{\vec{\ell}}^{-\mu+|\beta|_{\vec{\ell}}}\partial_x^\alpha\partial_{(\xi,\lambda)}^\beta a(x,\xi,\lambda);\ x \in \mathbb{R}^d,\ (\xi,\lambda) \in \mathbb{R}^d \times \Lambda\right\} \tag{3.13}$$

is $\mathcal{R}$-bounded in $\mathscr{L}(X, Y)$.

Proof. We have

$$S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y)) \cong S^0_{\mathrm{cl}}(\mathbb{R}^d_x)\hat{\otimes}_\pi S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y),$$

and thus we may write $a(x, \xi, \lambda) = \sum_{j=1}^\infty \lambda_j f_j(x) a_j(\xi, \lambda)$ with $(\lambda_j)_j \in \ell^1(\mathbb{N})$ and $f_j \to 0$ in S^0_{cl} and $a_j \to 0$ in $S^{\mu;\vec{\ell}}_{\mathrm{cl}}$.

Let $|\cdot|_{\alpha,\beta}$ be the seminorm of a symbol defined by the $\mathcal{R}$-bound of the set (3.13) (for any given function $a(x, \xi, \lambda)$). By Kahane's contraction principle (2.5) and Proposition 3.10 we obtain that for every α, β there exists a constant $C > 0$ and continuous seminorms q on S^0_{cl} and p on $S^{\mu;\vec{\ell}}_{\mathrm{cl}}$ such that $|\lambda_j f_j(x) a_j(\xi, \lambda)|_{\alpha,\beta} \le C|\lambda_j| q(f_j) p(a_j)$ for all $j \in \mathbb{N}$. Consequently, the series

$$\sum_{j=1}^\infty |\lambda_j f_j(x) a_j(\xi)|_{\alpha,\beta} < \infty$$

for all α, β which implies the assertion in view of Proposition 2.11. □

It follows from standard results in the theory of pseudodifferential operators that the class of operators we are considering is invariant under composition on rapidly decreasing functions, i.e., if X, Y, and Z are Banach spaces and

$$a(x, \xi, \lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu_1;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; Y, Z)),$$
$$b(x, \xi, \lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu_2;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y)),$$

then the composition $\mathrm{op}_x(a)(\lambda) \circ \mathrm{op}_x(b)(\lambda) : \mathscr{S}(\mathbb{R}^d, X) \to \mathscr{S}(\mathbb{R}^d, Z)$ is again a pseudodifferential operator $\mathrm{op}_x(a\#b)(\lambda)$ with symbol

$$(a\#b)(x, \xi, \lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu_1+\mu_2;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Z)).$$

The mapping $(a, b) \mapsto a\#b$ is bilinear and continuous in the symbol topology, and $a\#b \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \left(\partial_\xi^\alpha a\right) \left(D_x^\alpha b\right)$ in the sense that for every $K \in \mathbb{R}$ there is an $N(K)$ such that

$$a\#b - \sum_{|\alpha| \le N} \frac{1}{\alpha!} \left(\partial_\xi^\alpha a\right) \left(D_x^\alpha b\right) \in S^K(\mathbb{R}^d_x, S^{K;\vec{\ell}}(\mathbb{R}^d_\xi \times \Lambda; X, Z))$$

for all $N > N(K)$.

We are mainly interested in the mapping properties and dependence on the parameter $\lambda \in \Lambda$ of the operators $\mathrm{op}_x(a)(\lambda)$ in (anisotropic) vector valued Bessel potential spaces. To this end, recall the following

Definition 3.14. For $s \in \mathbb{R}$, $1 < p < \infty$, let $H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X)$ be the completion of $\mathscr{S}(\mathbb{R}^d, X)$ with respect to the norm

$$\|u\|_{H_p^{s;\vec{\ell}'}} := \|\,\mathrm{op}\left(\langle\xi\rangle^s_{\vec{\ell}'} \mathrm{Id}_X\right) u\|_{L_p(\mathbb{R}^d, X)}.$$

In order to proceed further we have to impose conditions on the Banach spaces involved ([1, 3, 4, 6, 12]):

Definition 3.15. A Banach space X

(i) is of *class* $(\mathcal{HT})$ if the Hilbert transform is continuous in $L_p(\mathbb{R}, X) \to L_p(\mathbb{R}, X)$ for some (all) $1 < p < \infty$. Recall that the Hilbert transform is the Fourier multiplier with symbol $\chi_{[0,\infty)}\mathrm{Id}_X$, where $\chi_{[0,\infty)}$ is the characteristic function on the interval $[0, \infty)$.
Banach spaces of class $(\mathcal{HT})$ are often called UMD-spaces, and it is worth mentioning that they are necessarily reflexive.

(ii) has Pisier's *property* (α) if for some $1 \le p < \infty$ there exists a constant $C_p > 0$ such that the inequality

$$\left(\sum_{\varepsilon_1,\dots,\varepsilon_N\in\{-1,1\}} \sum_{\varepsilon'_1,\dots,\varepsilon'_N\in\{-1,1\}} \left\| \sum_{j,k=1}^{N} \varepsilon_j\varepsilon'_k\alpha_{jk}x_{jk} \right\|^p \right)^{1/p}$$
$$\le C_p \left(\sum_{\varepsilon_1,\dots,\varepsilon_N\in\{-1,1\}} \sum_{\varepsilon'_1,\dots,\varepsilon'_N\in\{-1,1\}} \left\| \sum_{j,k=1}^{N} \varepsilon_j\varepsilon'_k x_{jk} \right\|^p \right)^{1/p}$$

holds for all choices $x_{jk} \in X$ and $\alpha_{jk} \in \{-1, 1\}$, $j, k = 1, \dots, N$, $N \in \mathbb{N}$.

Both $(\mathcal{HT})$ and (α) are purely topological properties of a Banach space and depend only on its isomorphism class. Some remarks about the permanence properties of these conditions are in order:

(a) Hilbert spaces are of class $(\mathcal{HT})$ and have property (α).
(b) (Finite) direct sums, closed subspaces, and quotients by closed subspaces of Banach spaces of class $(\mathcal{HT})$ are again of class $(\mathcal{HT})$.
(c) If X is of class $(\mathcal{HT})$ or has property (α), then so does $L_p(\Omega, X)$, $1 < p < \infty$, for any σ-finite measure space Ω.

Granted this, we immediately obtain a wealth of examples for Banach spaces of class $(\mathcal{HT})$ and property (α), most notably spaces of functions. We will generally assume henceforth that all Banach spaces are of class $(\mathcal{HT})$ and have property (α).

The following specialization of a Fourier multiplier theorem by Girardi and Weis is essential for us:

Theorem 3.16. ([6], Theorem 3.2) *Let X and Y be Banach spaces of class $(\mathcal{HT})$ with property (α). Let $\mathcal{T} \subset \mathscr{L}(X, Y)$ be an $\mathcal{R}$-bounded subset, and let*

$$\mathcal{M}(\mathcal{T}) := \{m \in C^d\left(\mathbb{R}^d \setminus \{0\}, \mathscr{L}(X, Y)\right);\ \xi^\beta \partial_\xi^\beta m(\xi) \in \mathcal{T}$$
$$\text{for } \xi \ne 0,\ \beta \le (1, \dots, 1)\}.$$

For $1 < p < \infty$ the set of associated Fourier multipliers

$$\left\{\mathcal{F}^{-1}_{\xi\to x} m(\xi)\mathcal{F}_{x\to\xi};\ m \in \mathcal{M}(\mathcal{T})\right\} \subset \mathscr{L}\left(L_p(\mathbb{R}^d, X), L_p(\mathbb{R}^d, Y)\right)$$

is then well defined (i.e., the Fourier multipliers are continuous in $L_p(\mathbb{R}^d, X) \to L_p(\mathbb{R}^d, Y)$*) and* $\mathcal{R}$*-bounded by* $C \cdot \mathcal{R}(\mathcal{T})$ *with a constant* $C \geq 0$ *depending only on* X, Y, p, *and the dimension* d.

Remark 3.17. From Theorem 3.16 we get in particular that the embedding

$$H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X) \hookrightarrow H_p^{t;\vec{\ell}'}(\mathbb{R}^d, X)$$

is well defined and continuous for $s \geq t$ and $1 < p < \infty$ for any Banach space X of class $(\mathcal{HT})$ satisfying property (α).

Moreover, for values $s \in \mathbb{N}$ such that $\ell'_j \mid s$ for all $j = 1, \ldots, d$, where $\vec{\ell}' = (\ell'_1, \ldots, \ell'_d) \in \mathbb{N}^d$, the space $H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X)$ coincides with the anisotropic Sobolev space

$$W_p^{s;\vec{\ell}'}(\mathbb{R}^d, X) = \left\{ u \in \mathscr{S}'(\mathbb{R}^d, X);\ \partial_x^\alpha u \in L_p(\mathbb{R}^d, X) \text{ for all } |\alpha|_{\vec{\ell}'} \leq s \right\}.$$

The latter follows by the usual argument: First, the Fourier multipliers $\mathcal{F}^{-1}_{\xi \to x} \xi^\alpha \cdot \mathrm{Id}_X \mathcal{F}_{x \to \xi}$ are continuous in $H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X) \to L_p(\mathbb{R}^d, X)$ for all $|\alpha|_{\vec{\ell}'} \leq s$, which shows the inclusion $H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X) \subset W_p^{s;\vec{\ell}'}(\mathbb{R}^d, X)$. On the other hand, there exist classical scalar symbols $\varphi_j \in S_{\mathrm{cl}}^{0;\vec{\ell}'}(\mathbb{R}^d_\xi)$, $j = 1, \ldots, d$, such that the function

$$m(\xi) = 1 + \sum_{j=1}^d \varphi_j(\xi) \xi_j^{s/\ell'_j} \in S_{\mathrm{cl}}^{s;\vec{\ell}'}(\mathbb{R}^d_\xi)$$

satisfies $m(\xi) \geq c \langle \xi \rangle_{\vec{\ell}'}^s$ for all $\xi \in \mathbb{R}^d$ with some constant $c > 0$, e.g., choose $\varphi_j(\xi) = \chi(\xi) |\xi|_{\vec{\ell}'}^{-s} \xi_j^{s/\ell'_j}$, where $\chi \in C^\infty(\mathbb{R}^d_\xi)$ is a suitable excision function of the origin. Now $\langle \xi \rangle_{\vec{\ell}'}^s = \langle \xi \rangle_{\vec{\ell}'}^s m(\xi)^{-1} m(\xi)$, and the Fourier multipliers

$$\mathcal{F}^{-1}_{\xi \to x} \left(\langle \xi \rangle_{\vec{\ell}'}^s m(\xi)^{-1} \cdot \mathrm{Id}_X \right) \mathcal{F}_{x \to \xi} : L_p(\mathbb{R}^d, X) \to L_p(\mathbb{R}^d, X),$$
$$\mathcal{F}^{-1}_{\xi \to x} m(\xi) \cdot \mathrm{Id}_X \mathcal{F}_{x \to \xi} : W_p^{s;\vec{\ell}'}(\mathbb{R}^d, X) \to L_p(\mathbb{R}^d, X)$$

are continuous. This shows that

$$\mathcal{F}^{-1}_{\xi \to x} \langle \xi \rangle_{\vec{\ell}'}^s \cdot \mathrm{Id}_X \mathcal{F}_{x \to \xi} : W_p^{s;\vec{\ell}'}(\mathbb{R}^d, X) \to L_p(\mathbb{R}^d, X)$$

is continuous, thus giving the other inclusion $W_p^{s;\vec{\ell}'}(\mathbb{R}^d, X) \subset H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X)$.

In the case $\dim X < \infty$ and $\dim Y < \infty$, Theorem 3.18 below strengthens classical results about norm estimates for pseudodifferential operators depending on parameters (see [16]) to the $\mathcal{R}$-boundedness of these families. It is that theorem combined with a parametrix construction in the calculus of pseudodifferential operators that yields the $\mathcal{R}$-boundedness of resolvents $\lambda(A - \lambda)^{-1}$ of anisotropic elliptic operators A in Sect. 4, and by a localization argument also of elliptic scattering operators in Sect. 5.

Theorem 3.18. *Let X and Y be Banach spaces of class $(\mathcal{HT})$ with property (α), and let $a(x,\xi,\lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y))$. Write $\vec{\ell} = (\vec{\ell}', \vec{\ell}'') \in \mathbb{N}^{d+q}$, and let $\nu \geq \mu$ be fixed.*

The family of pseudodifferential operators $\mathrm{op}_x(a)(\lambda) : \mathscr{S}(\mathbb{R}^d, X) \to \mathscr{S}(\mathbb{R}^d, Y)$ extends by continuity to

$$\mathrm{op}_x(a)(\lambda) : H^{s;\vec{\ell}'}_p(\mathbb{R}^d, X) \to H^{s-\nu;\vec{\ell}'}_p(\mathbb{R}^d, Y)$$

for every $s \in \mathbb{R}$ and $1 < p < \infty$, and the operator function

$$\Lambda \ni \lambda \mapsto \mathrm{op}_x(a)(\lambda) \in \mathscr{L}\left(H^{s;\vec{\ell}'}_p(\mathbb{R}^d, X), H^{s-\nu;\vec{\ell}'}_p(\mathbb{R}^d, Y)\right)$$

belongs to the $\mathcal{R}$-bounded symbol space $S^{\mu';\vec{\ell}''}_{\mathcal{R}}(\Lambda; H^{s;\vec{\ell}'}_p(\mathbb{R}^d, X), H^{s-\nu;\vec{\ell}'}_p(\mathbb{R}^d, Y))$ with $\mu' = \mu$ if $\nu \geq 0$, or $\mu' = \mu - \nu$ if $\nu < 0$.

The mapping $\mathrm{op}_x : a(x,\xi,\lambda) \mapsto \mathrm{op}_x(a)(\lambda)$ is continuous in the symbol spaces

$$S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y)) \to S^{\mu';\vec{\ell}''}_{\mathcal{R}}(\Lambda; H^{s;\vec{\ell}'}_p(\mathbb{R}^d, X), H^{s-\nu;\vec{\ell}'}_p(\mathbb{R}^d, Y)).$$

Proof. Let us begin with proving that

$$\mathrm{op}_x(a)(\lambda) : H^{s;\vec{\ell}'}_p(\mathbb{R}^d, X) \to H^{s-\nu;\vec{\ell}'}_p(\mathbb{R}^d, Y) \tag{3.19}$$

is continuous, and that the set $\{\langle\lambda\rangle^{-\mu'}_{\vec{\ell}''}\,\mathrm{op}_x(a)(\lambda);\ \lambda \in \Lambda\}$ is an $\mathcal{R}$-bounded subset of $\mathscr{L}\left(H^{s;\vec{\ell}'}_p(\mathbb{R}^d, X), H^{s-\nu;\vec{\ell}'}_p(\mathbb{R}^d, Y)\right)$ with $\mathcal{R}$-bound dominated by $C \cdot p(a)$ with a constant $C \geq 0$ not depending on $a(x,\xi,\lambda)$ and a continuous seminorm p on $S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y))$.

To this end note that

$$S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y)) \cong S^0_{\mathrm{cl}}(\mathbb{R}^d_x)\hat{\otimes}_\pi S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y),$$

hence it suffices to show this assertion for λ-dependent families of Fourier multipliers with symbols $a(\xi,\lambda) \in S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y)$, and to prove that multipliers

$$M(x)\mathrm{Id}_Y : H^{s-\nu;\vec{\ell}'}_p(\mathbb{R}^d, Y) \to H^{s-\nu;\vec{\ell}'}_p(\mathbb{R}^d, Y), \quad M(x) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x), \tag{3.20}$$

are continuous with continuous dependence on $M(x)$.

Let us consider the case of families of Fourier multipliers with symbols $a(\xi,\lambda)$, and assume first that $\mu = \nu = 0$. In view of $\langle D_x\rangle^s_{\vec{\ell}'}\,\mathrm{op}_x(a)(\lambda)\langle D_x\rangle^{-s}_{\vec{\ell}'} = \mathrm{op}_x(a)(\lambda)$ the desired assertion for arbitrary $s \in \mathbb{R}$ reduces to $s = 0$. Let

$$\mathcal{T} = \left\{\xi^\beta \partial^\beta_\xi a(\xi,\lambda);\ (\xi,\lambda) \in \mathbb{R}^d \times \Lambda,\ \beta \leq (1,\dots,1)\right\} \subset \mathscr{L}(X, Y).$$

The mapping

$$S^{0;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y) \to S^{0;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y), \quad a(\xi,\lambda) \mapsto \xi^\beta \partial^\beta_\xi a(\xi,\lambda)$$

is continuous, and thus the set $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathscr{L}(X, Y)$ by Proposition 3.10 by $\tilde{C} \cdot q(a)$ with some constant $\tilde{C} \geq 0$ not depending on $a(\xi, \lambda)$ and a continuous seminorm q on $S_{\mathrm{cl}}^{0;\vec{\ell}}(\mathbb{R}^d_\xi \times \Lambda; X, Y)$. Theorem 3.16 now gives the continuity of the Fourier multipliers $\mathrm{op}_x(a)(\lambda) : L_p(\mathbb{R}^d, X) \to L_p(\mathbb{R}^d, Y)$ and the $\mathcal{R}$-boundedness of the set

$$\{\mathrm{op}_x(a)(\lambda);\ \lambda \in \Lambda\} \subset \mathscr{L}(L_p(\mathbb{R}^d, X), L_p(\mathbb{R}^d, Y))$$

by a multiple of $q(a)$. This shows the assertion in the case $\mu = \nu = 0$.

Now consider the case of a general Fourier multiplier with symbol $a(\xi, \lambda) \in S_{\mathrm{cl}}^{\mu;\vec{\ell}}(\mathbb{R}^d_\xi \times \Lambda; X, Y)$. Writing $a(\xi, \lambda) = \langle \xi, \lambda\rangle_{\vec{\ell}}^{\mu} \cdot \Big(\langle \xi, \lambda\rangle_{\vec{\ell}}^{-\mu} a(\xi, \lambda)\Big)$ and noting that $a(\xi, \lambda) \mapsto \langle \xi, \lambda\rangle_{\vec{\ell}}^{-\mu} a(\xi, \lambda)$ is continuous in

$$S_{\mathrm{cl}}^{\mu;\vec{\ell}}(\mathbb{R}^d_\xi \times \Lambda; X, Y) \to S_{\mathrm{cl}}^{0;\vec{\ell}}(\mathbb{R}^d_\xi \times \Lambda; X, Y)$$

we obtain from the above that it is sufficient to show the $\mathcal{R}$-boundedness of the family

$$\langle \lambda\rangle_{\vec{\ell}''}^{-\mu'} \langle D_x, \lambda\rangle_{\vec{\ell}}^{\mu} : H_p^{s;\vec{\ell}'}(\mathbb{R}^d, Y) \to H_p^{s-\nu;\vec{\ell}'}(\mathbb{R}^d, Y),$$

i.e., the $\mathcal{R}$-boundedness of the λ-dependent family of Fourier multipliers with symbols $\langle \lambda\rangle_{\vec{\ell}''}^{-\mu'} \langle \xi, \lambda\rangle_{\vec{\ell}}^{\mu} \cdot \mathrm{Id}_Y$. Evidently, this further reduces to consider the family of Fourier multipliers with symbols

$$\langle \xi\rangle_{\vec{\ell}'}^{s-\nu} \langle \lambda\rangle_{\vec{\ell}''}^{-\mu'} \langle \xi, \lambda\rangle_{\vec{\ell}}^{\mu} \langle \xi\rangle_{\vec{\ell}'}^{-s} \cdot \mathrm{Id}_Y = \langle \xi\rangle_{\vec{\ell}'}^{-\nu} \langle \lambda\rangle_{\vec{\ell}''}^{-\mu'} \langle \xi, \lambda\rangle_{\vec{\ell}}^{\mu} \cdot \mathrm{Id}_Y$$

on $L_p(\mathbb{R}^d, Y)$. By Kahane's contraction principle (2.5) and Theorem 3.16 it is sufficient to prove that the function $\psi(\xi, \lambda) = \langle \xi\rangle_{\vec{\ell}'}^{-\nu} \langle \lambda\rangle_{\vec{\ell}''}^{-\mu'} \langle \xi, \lambda\rangle_{\vec{\ell}}^{\mu}$ satisfies

$$\sup\{|\xi^\beta \partial_\xi^\beta \psi(\xi, \lambda)|;\ (\xi, \lambda) \in \mathbb{R}^d \times \Lambda,\ \beta \leq (1, \ldots, 1)\} < \infty.$$

This, however, is an elementary estimate and follows easily.

We still have to consider the case of multipliers (3.20). Noting that

$$S_{\mathrm{cl}}^0(\mathbb{R}^d_x) \ni M(x) \mapsto \langle \xi\rangle_{\vec{\ell}'}^{s-\nu} \# M(x) \# \langle \xi\rangle_{\vec{\ell}'}^{-(s-\nu)} \in S_{\mathrm{cl}}^0(\mathbb{R}^d_x, S_{\mathrm{cl}}^{0;\vec{\ell}'}(\mathbb{R}^d_\xi))$$

is continuous, it suffices to show the continuity of pseudodifferential operators $\mathrm{op}_x\,(b(x, \xi) \cdot \mathrm{Id}_Y)$ on $L_p(\mathbb{R}^d, Y)$, where $b(x, \xi) \in S_{\mathrm{cl}}^0(\mathbb{R}^d_x, S_{\mathrm{cl}}^{0;\vec{\ell}'}(\mathbb{R}^d_\xi))$, and the operator norm of $\mathrm{op}_x\,(b(x, \xi) \cdot \mathrm{Id}_Y)$ has to be bounded by a multiple of a continuous seminorm of $b(x, \xi)$. As before write $S_{\mathrm{cl}}^0(\mathbb{R}^d_x, S_{\mathrm{cl}}^{0;\vec{\ell}'}(\mathbb{R}^d_\xi)) \cong S_{\mathrm{cl}}^0(\mathbb{R}^d_x) \hat{\otimes}_\pi S_{\mathrm{cl}}^{0;\vec{\ell}'}(\mathbb{R}^d_\xi)$, which reduces the assertion to Fourier multipliers with symbols in $S_{\mathrm{cl}}^{0;\vec{\ell}'}(\mathbb{R}^d_\xi) \cdot \mathrm{Id}_Y$ and multipliers with symbols in $S_{\mathrm{cl}}^0(\mathbb{R}^d_x) \cdot \mathrm{Id}_Y$ on $L_p(\mathbb{R}^d, Y)$. The case of Fourier multipliers follows from Theorem 3.16, and the case of multipliers on $L_p(\mathbb{R}^d, Y)$ is elementary.

It remains to show that (3.19) is indeed an $\mathcal{R}$-bounded symbol of (anisotropic) order $\mu' = \mu$ if $\nu \geq 0$ or $\mu' = \mu - \nu$ if $\nu < 0$, respectively, with symbol estimates dominated by a continuous seminorm of $a(x, \xi, \lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y))$. Note first that the mapping

$$\Lambda \ni \lambda \mapsto a(x, \xi, \lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu;\vec{\ell}'}_{\mathcal{R}}(\mathbb{R}^d_\xi; X, Y))$$

is C^∞ (with λ-derivatives being represented by those of the symbol a), and thus by what we have just proved we conclude that (3.19) depends smoothly on $\lambda \in \Lambda$, and

$$\partial^\beta_\lambda \operatorname{op}_x(a)(\lambda) = \operatorname{op}_x\left(\partial^\beta_\lambda a\right)(\lambda) : H^{s;\vec{\ell}'}_p(\mathbb{R}^d, X) \to H^{s-\nu;\vec{\ell}'}_p(\mathbb{R}^d, Y)$$

for $\beta \in \mathbb{N}^q_0$. Applying the above we obtain that $\{\langle\lambda\rangle^{-\mu'+|\beta|_{\vec{\ell}''}}_{\vec{\ell}''} \partial^\beta_\lambda \operatorname{op}_x(a)(\lambda);\ \lambda \in \Lambda\}$ is $\mathcal{R}$-bounded in $\mathscr{L}\left(H^{s;\vec{\ell}'}_p(\mathbb{R}^d, X), H^{s-\nu;\vec{\ell}'}_p(\mathbb{R}^d, Y)\right)$ by a multiple of a continuous seminorm of $\partial^\beta_\xi a(x, \xi, \lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu-|\beta|_{\vec{\ell}''};\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y))$. Since

$$\partial^\beta_\lambda : S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y)) \to S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu-|\beta|_{\vec{\ell}''};\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, Y))$$

is continuous the proof of the theorem is complete. □

4. Maximal regularity for anisotropic elliptic operators on $\mathbb{R}^d$

To illustrate how the results of the previous sections can be applied to prove maximal regularity results for partial differential operators, we consider in this section the case of anisotropic elliptic operators in $\mathbb{R}^d$.

Throughout this section let X be a Banach space of class $(\mathcal{HT})$ satisfying property (α). Let $\vec{\ell}' \in \mathbb{N}^d$ be a vector determining the anisotropy, and let

$$A = \sum_{|\alpha|_{\vec{\ell}'} \leq \mu} a_\alpha(x) D^\alpha_x : \mathscr{S}(\mathbb{R}^d, X) \to \mathscr{S}(\mathbb{R}^d, X), \tag{4.1}$$

where the $a_\alpha(x) \in S^0_{\mathrm{cl}}(\mathbb{R}^d, \mathscr{L}(X))$ are operator valued coefficient functions, and $\mu \in \mathbb{N}$.

Let $\Lambda \subset \mathbb{C}$ be a closed sector. We assume the following anisotropic ellipticity condition of A with respect to Λ:

Definition 4.2. A is called *parameter-dependent anisotropic elliptic with respect to* Λ if the following two conditions are fulfilled:

(i) The spectrum of the *(anisotropic) principal symbol*

$$\sum_{|\alpha|_{\vec{\ell}'} = \mu} a_\alpha(x)\xi^\alpha \in \mathscr{L}(X)$$

intersected with Λ is empty for all $\xi \in \mathbb{R}^d \setminus \{0\}$ and all $x \in \mathbb{R}^d$.

(ii) The spectrum of the *extended principal symbol*

$$\sum_{|\alpha|_{\vec{\ell}'}=\mu} a_{\alpha,(0)}(x)\xi^\alpha \in \mathscr{L}(X)$$

intersected with Λ is empty for all $\xi \in \mathbb{R}^d \setminus \{0\}$ and all $x \in \mathbb{R}^d \setminus \{0\}$, where $a_{\alpha,(0)}(x)$ is the principal component of the $\mathscr{L}(X)$-valued classical symbol $a_\alpha(x) \in S^0_{\mathrm{cl}}(\mathbb{R}^d, \mathscr{L}(X))$.

The extended principal symbol (on $|x| = 1$) can be regarded as an extension of the anisotropic principal symbol to the radial compactification of $\mathbb{R}^d$ (in the x-variables).

By the remarks given in the introduction, maximal regularity for anisotropic elliptic operators A (up to a spectral shift) follows from Theorem 4.3 below provided that A is parameter-dependent anisotropic elliptic with respect to the right half-plane $\Lambda = \{\lambda \in \mathbb{C};\ \Re(\lambda) \geq 0\} \subset \mathbb{C}$, see Corollary 4.5. Note that the Sobolev spaces $H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X)$ are of class $(\mathcal{HT})$ and satisfy property (α) in view of the permanence properties of these conditions since they are isomorphic to $L_p(\mathbb{R}^d, X)$.

Theorem 4.3. *Let A be parameter-dependent anisotropic elliptic with respect to the closed sector $\Lambda \subset \mathbb{C}$. Then*

$$A : H_p^{s+\mu;\vec{\ell}'}(\mathbb{R}^d, X) \to H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X) \tag{4.4}$$

is continuous for every $s \in \mathbb{R}$ and $1 < p < \infty$, and A with domain $H_p^{s+\mu;\vec{\ell}'}(\mathbb{R}^d, X)$ is a closed operator in $H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X)$.

For $\lambda \in \Lambda$ with $|\lambda| \geq R$ sufficiently large the operator

$$A - \lambda : H_p^{s+\mu;\vec{\ell}'}(\mathbb{R}^d, X) \to H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X)$$

is invertible for all $s \in \mathbb{R}$, and the resolvent

$$\{\lambda(A-\lambda)^{-1};\ \lambda \in \Lambda,\ |\lambda| \geq R\} \subset \mathscr{L}\left(H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X)\right)$$

is $\mathcal{R}$-bounded.

Proof. The continuity of (4.4) follows from Theorem 3.18.

The operator $A - \lambda : \mathscr{S}(\mathbb{R}^d, X) \to \mathscr{S}(\mathbb{R}^d, X)$ is of the form $\mathrm{op}_x(a)(\lambda)$ with the symbol

$$a(x, \xi, \lambda) = \sum_{|\alpha|_{\vec{\ell}'} \leq \mu} a_\alpha(x)\xi^\alpha - \lambda \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, X)),$$

where $\vec{\ell} = (\vec{\ell}', \mu, \mu) \in \mathbb{N}^{d+2}$. Note that $\Lambda \subset \mathbb{C} \cong \mathbb{R}^2$ is considered as a real 2-dimensional parameter space.

By our assumption of parameter-dependent anisotropic ellipticity, the parameter-dependent principal symbol

$$a_{(\mu)}(x,\xi,\lambda) = \sum_{|\alpha|_{\vec{\ell}'}=\mu} a_\alpha(x)\xi^\alpha - \lambda \in \mathscr{L}(X)$$

is invertible for all $x \in \mathbb{R}^d$ and $(\xi,\lambda) \in \left(\mathbb{R}^d \times \Lambda\right) \setminus \{0\}$, and with any excision function $\chi \in C^\infty(\mathbb{R}^{d+2})$ of the origin (i.e., $\chi \equiv 0$ near the origin and $\chi \equiv 1$ near infinity) we have

$$b(x,\xi,\lambda) = \chi(\xi,\lambda)a_{(\mu)}(x,\xi,\lambda)^{-1} \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{-\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, X)).$$

We conclude that

$$a\#b - 1,\ b\#a - 1 \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{-1;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, X)),$$

and the standard formal Neumann series argument now implies the existence of

$$p(x,\xi,\lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{-\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\xi \times \Lambda; X, X)),$$
$$r_j(x,\xi,\lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_x, S^{-\infty}(\mathbb{R}^d_\xi \times \Lambda; X, X)), \quad j = 1, 2,$$

such that $a\#p = 1 + r_1$ and $p\#a = 1 + r_2$.

Let $P(\lambda) = \mathrm{op}_x(p)(\lambda)$, and $R_j(\lambda) = \mathrm{op}_x(r_j)(\lambda)$, $j = 1, 2$. By Theorem 3.18 we have

$$P(\lambda) \in S^{0;(\mu,\mu)}_{\mathcal{R}}\left(\Lambda; H^{s;\vec{\ell}'}_p, H^{s+\mu;\vec{\ell}'}_p\right) \cap S^{-\mu;(\mu,\mu)}_{\mathcal{R}}\left(\Lambda; H^{s;\vec{\ell}'}_p, H^{s;\vec{\ell}'}_p\right),$$
$$R_j(\lambda) \in \mathscr{S}\left(\Lambda, \mathscr{L}\left(H^{s;\vec{\ell}'}_p, H^{t;\vec{\ell}'}_p\right)\right), \quad j = 1, 2,$$

for all $s, t \in \mathbb{R}$. For $\lambda \in \Lambda$ with $|\lambda| \geq R$, the operators

$$1 + R_j(\lambda) : H^{t;\vec{\ell}'}_p(\mathbb{R}^d, X) \to H^{t;\vec{\ell}'}_p(\mathbb{R}^d, X)$$

are invertible for every $t \in \mathbb{R}$, and the inverses are represented as $1 + R'_j(\lambda)$ with

$$R'_j(\lambda) \in \mathscr{S}\left(\Lambda, \mathscr{L}\left(H^{s;\vec{\ell}'}_p(\mathbb{R}^d, X), H^{s';\vec{\ell}'}_p(\mathbb{R}^d, X)\right)\right), \quad j = 1, 2,$$

for any $s, s' \in \mathbb{R}$. More precisely, we may write

$$R'_j(\lambda) = -R_j(\lambda) + R_j(\lambda)\chi(\lambda)\left(1 + R_j(\lambda)\right)^{-1} R_j(\lambda)$$

with some excision function $\chi \in C^\infty(\mathbb{R}^2)$ of the origin, and $\left(1 + R_j(\lambda)\right)^{-1}$ is the inverse of $1 + R_j(\lambda)$ on some space $H^{t_0;\vec{\ell}'}_p(\mathbb{R}^d, X)$ with a fixed $t_0 \in \mathbb{R}$.

We obtain that for $\lambda \in \Lambda$ with $|\lambda| \geq R$ the operator

$$A - \lambda : H^{s+\mu;\vec{\ell}'}_p(\mathbb{R}^d, X) \to H^{s;\vec{\ell}'}_p(\mathbb{R}^d, X)$$

is invertible for every $s \in \mathbb{R}$, and the resolvent is represented as

$$(A - \lambda)^{-1} = P(\lambda) + P(\lambda)R'_1(\lambda).$$

In view of Theorem 3.18 and Corollary 2.14 the proof is therefore complete. □

Corollary 4.5. *Let $1 < p < \infty$, and assume that A is parameter-dependent anisotropic elliptic with respect to $\Lambda = \{\lambda \in \mathbb{C};\ \Re(\lambda) \geq 0\}$. Then there exists $\gamma \in \mathbb{R}$ such that $A+\gamma$ with domain $H_p^{s+\mu;\vec{\ell}'}(\mathbb{R}^d, X) \subset H_p^{s;\vec{\ell}'}(\mathbb{R}^d, X)$ has maximal regularity for every $s \in \mathbb{R}$.*

5. Maximal regularity for elliptic scattering operators

Let $\overline{M}$ be a d-dimensional smooth compact manifold with boundary. The aim of this section is to prove maximal regularity for elliptic scattering operators on $\overline{M}$. These are elliptic differential operators in the interior $M = \mathring{\overline{M}}$ which degenerate at the boundary in a specific way. The model example in this context is $\overline{M} = \mathbb{B}$, a d-dimensional ball. In this case an elliptic scattering operator on $\mathbb{B}$ corresponds to an elliptic operator on $\mathbb{R}^d$ whose coefficients behave in some nice way when $|x| \to \infty$ radially, the ball $\mathbb{B}$ appears as the radial compactification of $\mathbb{R}^d$.

We follow Melrose [11] with our presentation of scattering operators. The presence of coefficients in the morphisms of a bundle of (infinite dimensional) Banach spaces represents no major difficulty as far as it concerns the action of the operators on smooth sections that vanish to infinite order at the boundary of $\overline{M}$.

Scattering differential operators. Let $\mathbf{x}$ be a defining function for the boundary of $\overline{M}$, i.e., $\mathbf{x} \in C^\infty(\overline{M})$ with $\mathbf{x} > 0$ on M, $\mathbf{x} = 0$ on $\partial\overline{M}$, and $d\mathbf{x} \neq 0$ on $\partial\overline{M}$. Let ${}^{\mathrm{b}}\mathcal{V}(\overline{M})$ denote the vector fields on $\overline{M}$ which are tangent on $\partial\overline{M}$, and let

$${}^{\mathrm{sc}}\mathcal{V}(\overline{M}) = \mathbf{x}\,{}^{\mathrm{b}}\mathcal{V}(\overline{M})$$

be the Lie algebra of *scattering vector fields* on $\overline{M}$. In coordinates near $\partial\overline{M}$, the vector fields in ${}^{\mathrm{sc}}\mathcal{V}(\overline{M})$ are spanned by

$$x^2\frac{\partial}{\partial x},\ x\frac{\partial}{\partial y_j},\quad j = 1,\ldots,d-1, \tag{5.1}$$

where $y_1,\ldots,y_{d-1}$ are coordinates on $\partial\overline{M}$ and x is a local boundary defining function.

Let ${}^{\mathrm{sc}}T(\overline{M}) \to \overline{M}$ denote the *scattering tangent bundle*, i.e., the vector bundle on $\overline{M}$ whose sections are the scattering vector fields. Fibrewise, we may represent ${}^{\mathrm{sc}}T(\overline{M})$ as

$${}^{\mathrm{sc}}T_p(\overline{M}) = {}^{\mathrm{sc}}\mathcal{V}(\overline{M})/\mathcal{I}_p(\overline{M}) \cdot {}^{\mathrm{sc}}\mathcal{V}(\overline{M}),$$

where $\mathcal{I}_p(\overline{M}) \subset C^\infty(\overline{M})$ is the ideal of functions that vanish at p. Locally near $\partial\overline{M}$, the vector fields (5.1) are a frame for ${}^{\mathrm{sc}}T(\overline{M})$. Let ${}^{\mathrm{sc}}T^*(\overline{M}) \to \overline{M}$ be the *scattering cotangent bundle*, the dual of ${}^{\mathrm{sc}}T(\overline{M})$.

Let ${}^{\mathrm{sc}}\mathrm{Diff}^*(\overline{M})$ be the enveloping algebra generated by ${}^{\mathrm{sc}}\mathcal{V}(\overline{M})$ and $C^\infty(\overline{M})$ consisting of the *scattering differential operators*. The operators of order $\mu \in \mathbb{N}_0$ are denoted as usual by ${}^{\mathrm{sc}}\mathrm{Diff}^\mu(\overline{M})$. The principal symbol $\boldsymbol{\sigma}(A)$ on $T^*M \setminus 0$ of an operator $A \in {}^{\mathrm{sc}}\mathrm{Diff}^\mu(\overline{M})$ lifts to a well defined homogeneous function ${}^{\mathrm{sc}}\boldsymbol{\sigma}(A)$

of degree μ on ${}^{\mathrm{sc}}T^*\overline{M} \setminus 0$ which is called the *principal scattering* or *principal* sc-*symbol* of A. Note that ${}^{\mathrm{sc}}\boldsymbol{\sigma}(A)$ is different from the symbol map considered in Sect. 6 of [11], we are not heading for a Fredholm theory here. Instead, ${}^{\mathrm{sc}}\boldsymbol{\sigma}(A)$ may somewhat be regarded as a unified version of the principal symbol and the extended principal symbol from Definition 4.2 (when $\overline{M} = \mathbb{B}$ is a ball).

We are interested in maximal regularity of elliptic scattering operators with operator valued coefficients. To this end, let $X \to \overline{M}$ be a smooth vector bundle of Banach spaces with typical fibre X_0 that is of class $(\mathcal{HT})$ and satisfies property (α) (X is the restriction of a corresponding bundle of Banach spaces with these properties from a neighboring smooth manifold without boundary to $\overline{M}$, e.g., from the double $2\overline{M}$). Note that both $(\mathcal{HT})$ and property (α) are topological properties of a Banach space, hence these notions are well defined for bundles of Banach spaces.

Let ${}^{\mathrm{sc}}\mathrm{Diff}^{\mu}(\overline{M}; X)$ be the space of scattering differential operators

$$A : \dot{C}^{\infty}(\overline{M}, X) \to \dot{C}^{\infty}(\overline{M}, X)$$

of order μ with coefficients in the bundle $\mathscr{L}(X)$ of bounded operators in the fibres of X. A has a principal sc-symbol

$${}^{\mathrm{sc}}\boldsymbol{\sigma}(A) \in C^{\infty}\left({}^{\mathrm{sc}}T^*\overline{M} \setminus 0, \mathscr{L}({}^{\mathrm{sc}}\pi^* X)\right),$$

where ${}^{\mathrm{sc}}\pi : {}^{\mathrm{sc}}T^*\overline{M} \setminus 0 \to \overline{M}$ is the canonical projection.

Function spaces and pseudodifferential operators. Before coming to a general manifold $\overline{M}$, we consider first the special case

$$\mathbb{S}^d_+ = \{z' = (z'_1, \ldots, z'_{d+1}) \in \mathbb{R}^{d+1};\ |z'| = 1,\ z'_1 \geq 0\}.$$

Consider the stereographic projection

$$\mathrm{SP} : \mathbb{R}^d \ni z \longmapsto \left(\frac{1}{(1+|z|^2)^{1/2}}, \frac{z}{(1+|z|^2)^{1/2}}\right) \in \mathbb{S}^d_+.$$

For a Banach space X_0 of class $(\mathcal{HT})$ having property (α) and $s \in \mathbb{R}$, $1 < p < \infty$, the vector valued sc-Sobolev space on $\mathbb{S}^d_+$ is defined as

$${}^{\mathrm{sc}}H^s_p(\mathbb{S}^d_+, X_0) = \mathrm{SP}_*\, H^s_p(\mathbb{R}^d, X_0).$$

Moreover, for a closed sector $\Lambda \subset \mathbb{C}$ and any $\ell \in \mathbb{N}$, we define the class ${}^{\mathrm{sc}}\Psi^{\mu;\ell}(\mathbb{S}^d_+; \Lambda)$ as to consist of families of operators

$$A(\lambda) : \dot{C}^{\infty}(\mathbb{S}^d_+, X_0) \to \dot{C}^{\infty}(\mathbb{S}^d_+, X_0), \quad \lambda \in \Lambda,$$

such that

$$\left(\mathrm{SP}^* A(\lambda)\right) u(z) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{iz\zeta} a(z, \zeta, \lambda)\hat{u}(\zeta)\, d\zeta, \quad u \in \mathscr{S}(\mathbb{R}^d, X_0),$$

is a pseudodifferential operator with symbol

$$a(z, \zeta, \lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_z, S^{\mu;\vec{\ell}}_{\mathrm{cl}}(\mathbb{R}^d_\zeta \times \Lambda; X_0, X_0)),$$

where the vector $\vec{\ell}$ that determines the anisotropy of covariables ζ and parameters λ is given by $\vec{\ell} = (\underbrace{1, \ldots, 1}_{d}, \ell, \ell) \in \mathbb{N}^{d+2}$. Note that $\Lambda \subset \mathbb{C} \cong \mathbb{R}^2$ is regarded as a real 2-dimensional parameter space.

In the general case of a compact manifold with boundary $\overline{M}$ and a bundle $X \to \overline{M}$ of Banach spaces of class $(\mathcal{HT})$ having property (α), we define ${}^{\mathrm{sc}}H^s_p(\overline{M}, X)$ as to consist of all $u \in \mathcal{D}'(M, X)$ such that $\chi_*(\varphi u) \in {}^{\mathrm{sc}}H^s_p(\mathbb{S}^d_+, X_0)$ for all local charts $\chi : U \to \mathbb{S}^d_+$ with $X|_U \cong U \times X_0$, and all $\varphi \in C^\infty(\overline{M})$ with compact support contained in U. The invariance of scattering pseudodifferential operators (see also below) and Theorem 3.18 (a version of that theorem without parameters is sufficient) imply that the spaces ${}^{\mathrm{sc}}H^s_p(\overline{M}, X)$ are well defined for every $s \in \mathbb{R}$ and $1 < p < \infty$. Moreover, the projective topology with respect to the mappings

$$ {}^{\mathrm{sc}}H^s_p(\overline{M}, X) \ni u \mapsto \chi_*(\varphi u) \in {}^{\mathrm{sc}}H^s_p(\mathbb{S}^d_+, X_0) $$

for all charts χ and cut off functions φ (as well as trivializations of X) makes ${}^{\mathrm{sc}}H^s_p(\overline{M}, X)$ a topological vector space which is normable so as to be a Banach space that contains $\dot{C}^\infty(\overline{M}, X)$ as dense subspace.

The class ${}^{\mathrm{sc}}\Psi^{\mu;\ell}(\overline{M}; \Lambda)$ of (anisotropic) parameter-dependent scattering pseudodifferential operators on $\overline{M}$ with coefficients in $\mathscr{L}(X)$ consists of operator families

$$ A(\lambda) : \dot{C}^\infty(\overline{M}, X) \to \dot{C}^\infty(\overline{M}, X), \quad \lambda \in \Lambda, \tag{5.2} $$

such that the following holds:

- For all $\varphi, \psi \in C^\infty(\overline{M})$ with disjoint supports we have

$$ (\varphi A(\lambda)\psi)\, u(z) = \int_M k(z, z')u(z')\, \mathfrak{m}(z'), \quad u \in \dot{C}^\infty(\overline{M}, X), $$

  with $k(z, z') \in \dot{C}^\infty\left(\overline{M} \times \overline{M}, \mathscr{L}(\pi_L^* X, \pi_R^* X)\right)$, where $\pi_L, \pi_R : \overline{M} \times \overline{M} \to \overline{M}$ are the canonical projections on the left and right factor, respectively, and $\mathfrak{m}$ is any scattering density, i.e., $\mathbf{x}^{d+1}\mathfrak{m}$ is a smooth everywhere positive density on $\overline{M}$ (recall that $d = \dim \overline{M}$).
- For any chart $\chi : U \to \mathbb{S}^d_+, U \subset \overline{M}$, with $X|_U \cong U \times X_0$, and all $\varphi, \psi \in C^\infty(\overline{M})$ with compact supports contained in U, the operator push-forward $\chi_*(\varphi A(\lambda)\psi)$ is required to belong to the class ${}^{\mathrm{sc}}\Psi^{\mu;\ell}(\mathbb{S}^d_+; \Lambda)$ as defined above.

A localization argument and Theorem 3.18 now imply the following

Theorem 5.3. *Let* $A(\lambda) \in {}^{\mathrm{sc}}\Psi^{\mu;\ell}(\overline{M}; \Lambda)$. *Then* (5.2) *extends by continuity to a family of continuous operators*

$$ A(\lambda) : {}^{\mathrm{sc}}H^s_p(\overline{M}, X) \to {}^{\mathrm{sc}}H^{s-\nu}_p(\overline{M}, X) $$

for every $s \in \mathbb{R}$ *and all* $1 < p < \infty$, $\nu \geq \mu$. *The operator function*

$$ \Lambda \ni \lambda \mapsto A(\lambda) \in \mathscr{L}\left({}^{\mathrm{sc}}H^s_p(\overline{M}, X), {}^{\mathrm{sc}}H^{s-\nu}_p(\overline{M}, X)\right) $$

belongs to the $\mathcal{R}$*-bounded symbol space* $S^{\mu';(\ell,\ell)}_{\mathcal{R}}\left(\Lambda; {}^{\mathrm{sc}}H^s_p(\overline{M}, X), {}^{\mathrm{sc}}H^{s-\nu}_p(\overline{M}, X)\right)$ *with* $\mu' = \mu$ *if* $\nu \geq 0$, *or* $\mu' = \mu - \nu$ *if* $\nu < 0$.

$\mathcal{R}$-boundedness of resolvents. We are now ready to prove the $\mathcal{R}$-boundedness of resolvents of elliptic scattering differential operators with operator valued coefficients. As mentioned in the introduction, maximal regularity (up to a spectral shift) is a consequence if we choose $\Lambda \subset \mathbb{C}$ to be the right half-plane in Theorem 5.4 below, see Corollary 5.6. Note that the sc-Sobolev spaces are of class $(\mathcal{HT})$ and satisfy property (α) as they are isomorphic to a finite direct sum of X_0-valued L_p-spaces, and X_0 has these properties.

Theorem 5.4. *Let $\Lambda \subset \mathbb{C}$ be a closed sector, and let $A \in {}^{\mathrm{sc}}\mathrm{Diff}^{\mu}(\overline{M}; X)$, $\mu > 0$, be a scattering differential operator on $\overline{M}$ with coefficients in $\mathscr{L}(X)$, where $X \to \overline{M}$ is a smooth vector bundle of Banach spaces of class $(\mathcal{HT})$ satisfying property (α). Assume that*

$$\operatorname{spec}\left({}^{\mathrm{sc}}\boldsymbol{\sigma}(A)(z,\zeta)\right) \cap \Lambda = \emptyset$$

for all $(z,\zeta) \in {}^{\mathrm{sc}}T^*\overline{M} \setminus 0$.

Then $A : {}^{\mathrm{sc}}H_p^{s+\mu}(\overline{M}, X) \to {}^{\mathrm{sc}}H_p^{s}(\overline{M}, X)$ is continuous for every $s \in \mathbb{R}$ and $1 < p < \infty$, and A with domain ${}^{\mathrm{sc}}H_p^{s+\mu}(\overline{M}, X)$ is a closed operator in ${}^{\mathrm{sc}}H_p^{s}(\overline{M}, X)$.

Moreover, for $\lambda \in \Lambda$ with $|\lambda| \geq R$ sufficiently large, the operator

$$A - \lambda : {}^{\mathrm{sc}}H_p^{s+\mu}(\overline{M}, X) \to {}^{\mathrm{sc}}H_p^{s}(\overline{M}, X) \tag{5.5}$$

is invertible for all $s \in \mathbb{R}$, and the resolvent

$$\{\lambda(A-\lambda)^{-1};\ \lambda \in \Lambda,\ |\lambda| \geq R\} \subset \mathscr{L}\left({}^{\mathrm{sc}}H_p^{s}(\overline{M}, X)\right)$$

is $\mathcal{R}$-bounded.

Proof. In view of the parameter-dependent ellipticity condition on the principal sc-symbol of A we conclude that we can construct local parametrices as in the proof of Theorem 4.3 by symbolic inversion and a formal Neumann series argument. Patching these parametrices together on $\overline{M}$ with a partition of unity gives a global parameter-dependent parametrix $P(\lambda) \in {}^{\mathrm{sc}}\Psi^{-\mu;\mu}(\overline{M}; \Lambda)$ of $A - \lambda$, i.e.,

$$(A-\lambda)\,P(\lambda) - 1,\ (A-\lambda)\,P(\lambda) - 1 \in {}^{\mathrm{sc}}\Psi^{-\infty;\mu}(\overline{M}; \Lambda).$$

By Theorem 5.3 we hence conclude that (5.5) is invertible for all $s \in \mathbb{R}$ and $|\lambda| \geq R$ sufficiently large, and for these λ we may write

$$(A-\lambda)^{-1} - P(\lambda) \in \mathscr{S}\left(\Lambda, \mathscr{L}({}^{\mathrm{sc}}H_p^{s}(\overline{M}, X), {}^{\mathrm{sc}}H_p^{t}(\overline{M}, X))\right)$$

for all $s, t \in \mathbb{R}$. Theorem 5.3 and Corollary 2.14 now imply the assertion. □

Corollary 5.6. *Let $1 < p < \infty$, and let $A \in {}^{\mathrm{sc}}\mathrm{Diff}^{\mu}(\overline{M}; X)$, $\mu > 0$. We assume that*

$$\operatorname{spec}\left({}^{\mathrm{sc}}\boldsymbol{\sigma}(A)(z,\zeta)\right) \cap \{\lambda \in \mathbb{C};\ \Re(\lambda) \geq 0\} = \emptyset$$

for all $(z,\zeta) \in {}^{\mathrm{sc}}T^*\overline{M} \setminus 0$. Then there exists $\gamma \in \mathbb{R}$ such that $A + \gamma$ with domain ${}^{\mathrm{sc}}H_p^{s+\mu}(\overline{M}, X) \subset {}^{\mathrm{sc}}H_p^{s}(\overline{M}, X)$ has maximal regularity for every $s \in \mathbb{R}$.

Acknowledgments. We would like to express our gratitude to Jan Prüss from the University of Halle for several discussions, in particular for bringing to our attention the paper by Girardi and Weis [6].

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